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NOTE

ON A THEOREM OF FRAENKEL, LEVITT AND SHIMSHONI

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Let α be an irrational number satisfying

$$\alpha > 1. \tag{1}$$

In 1972 Fraenkel, Levitt and Shimshoni [1] gave a characterization of those natural numbers K which occur in the sequence

$$[n\alpha] \quad (n = 1, 2, \ldots). \tag{2}$$

In the present short note I give another characterization which is in my opinion essentially simpler. Namely, while in the characterization proved in the present paper one has to verify an inequality on the fractional part of K/α, in that of [1] one has to know the parity of the first non-vanishing coefficient in a certain representation of K, where the denominators of the convergents of the continued fraction expansion of α occur. If α is already given, then one can calculate K/α immediately. On the other hand to apply the criterion of [1] one has to find the representation of K in terms of the Ostrowski-algorithm for which the knowledge of the regular continued fraction expansion of α is necessary. To find this representation, a greater calculating effort is necessary.

The referee of the present paper kindly called my attention to the paper [2] of Graham, Lin and Lin, where a somewhat different characterization is given. Of course, the characterizations of [1], [2] and the present paper are equivalent. I give a direct proof of the fact that my characterization and that of [1] are equivalent.

Theorem. *The necessary and sufficient condition for K to be contained in the sequence* (2) *is*

$$1 - 1/\alpha < \{K/\alpha\} \tag{3}$$

where $\{\cdots\}$ *denotes the fractional part.*

Proof. Suppose that K is contained in the sequence (2). Then the corresponding n

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has to be

$$n=[K/\alpha]+1, \tag{4}$$

namely, $[\alpha[K/\alpha]]=[K-\alpha\{K/\alpha\}]<K$ and $[\alpha([K/\alpha]+2)]=K+[\alpha(2-\{K/\alpha\})]\geq K+1$. Now we have

$$[\alpha[1+K/\alpha]]=K+[\alpha(1-\{K/\alpha\})],$$

but

$$[\alpha(1-\{K/\alpha\})]=0 \text{ iff } \{K/\alpha\}>1-1/\alpha,$$

which proves our theorem. □

As an application, I give an alternative proof of another result of [1], according to which for any irrational α satisfying $1<\alpha<2$ and $\beta=\alpha/(\alpha-1)$, any natural number K belongs to exactly one of the sequences

$$S_1=[n\alpha] \quad (n=1,2,\ldots)$$

and

$$S_2=[n\beta] \quad (n=1,2,\ldots).$$

Suppose that $K\in S_1$, so that $\{K/\alpha\}>1-1/\alpha$. Then

$$\{K(\alpha-1)/\alpha\}<1-(\alpha-1)/\alpha=1/\alpha;$$

namely,

$$\{K(\alpha-1)/\alpha\}=\{K-K/\alpha\}=1-\{K/\alpha\}<1/\alpha.$$

Similarly, $\{K/\alpha\}<1-1/\alpha$ implies $\{K(\alpha-1)/\alpha\}>1/\alpha$, which proves our statement.

After submitting the present paper I found that the above result was given as Problem 23 on p. 84 in the well-known book on number-theory [3] by Niven and Zuckerman. Since the problem was given among problems concerning the [ ]-function, it is to be assumed that the authors had in mind a solution similar to that given above.

Finally, I show directly that the characterization of our theorem is equivalent to that of [1]. Denote by

$$1/\alpha=[0;a_1,a_2,\ldots] \tag{5}$$

the regular continued fraction expansion of $1/\alpha$. Set

$$[0;a_1,\ldots,a_k]=A_k/B_k, \quad \text{where } (A_k,B_k)=1, \tag{6}$$

$$[a_k;a_{k+1},\ldots]=\zeta_k, \tag{7}$$

and

$$D_k=B_k/\alpha-A_k=(-1)^k/(B_k\zeta_{k+1}+B_{k-1}) \tag{8}$$

(see Perron [5, p. 39]).

As is known (see Ostrowski [4]), every natural number K admits a unique representation

$$K = \sum_{k=0}^{r} c_{k+1} B_k \tag{9}$$

where $0 \leqslant c_1 < a_1$, $0 \leqslant c_{k+1} \leqslant a_{k+i}$ and $c_{k+1} = a_{k+i}$ implies that $c_k = 0$.

Set

$$K/\alpha - L = \sum_{k=0}^{r} c_{k+i} D_k. \tag{10}$$

It follows from (8) and the recursion formula $D_{k+1} = a_{k+1} D_k + D_{k-1}$ that

$$-1/\alpha < K/\alpha - L < 1 - 1/\alpha; \tag{11}$$

namely,

$$\sum_{k=0}^{r} c_{k+1} D_k > a_2 D_1 + a_4 D_3 + \cdots = D_0 = -1/\alpha$$

and

$$\sum_{k=0}^{r} c_{k+1} D_k < (a_1 - 1) D_0 + a_3 D_2 + \cdots$$
$$< D_{-1} - D_0 = 1 - 1/\alpha,$$

where $D_{-1} = B_{-1}/\alpha - A_{-1}$, $B_{-1} = 0$ and $A_{-1} = 1$ (see Perron [5, p. 29]).

Now if the first non-vanishing coefficient c_{k+1} in (9) is with an even k, then $K/\alpha - L$ is positive and

$$K/\alpha - L = \{K/\alpha\};$$

therefore, because of (11), (3) cannot hold. If the first non-vanishing coefficient c_{k+1} is for an odd k, then $-1 < K\alpha - L < 0$ and so

$$1 + (K/\alpha - L) = \{K/\alpha\}.$$

Hence (11) yields that (3) must hold. Thus (3) holds if and only if the first non-vanishing coefficent c_{k+1} is for an odd k, which is the criterion of [1].

References

[1] A.S. Fraenkel, J. Levitt and M. Shimshoni, Characterization of the set of values $f(n) = [n\alpha]$, $n = 1, 2, \ldots$, Discrete Math. 2 (1972) 335–345.

[2] L. Graham, S. Lin and C.S. Lin, Spectra of numbers, Math. Mag. 51 (1978) 174–176.

[3] I. Niven and H.S. Zucerman, An Introduction to the Theory of Numbers (Wiley, New York, 3rd ed., 1972).

[4] A. Ostrowski, Bemerkungen zur Theorie der Diophantischen Approximation, Hamb. Abh. 1 (1921) 77–98.

[5] O. Perron, Die Lehre von den Kettenbruchen II (Aufloge, Leipzig, 1929).